

SPIN MAGNETOHYDRODYNAMICS

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Abstract

Starting from the non-relativistic Pauli description of spin $\frac{1}{2}$ particles, a set of fluid equations, governing the dynamics of such particles interacting with external fields and other particles, is derived. The equations describe electrons, positrons, holes, and similar conglomerates. In the case of electrons, the magnetohydrodynamic limit of an electron-ion plasma is investigated. The results should be of interest and relevance both to laboratory and astrophysical plasmas.

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1 Introduction

The concept of a magnetoplasma has attracted interest ever since first introduced by Alfvén [1], who showed the existence of waves in magnetized plasmas. Since then, magnetohydrodynamics (MHD) has grown into a vast and mature field of science, with applications ranging from solar physics and astrophysical dynamos, to fusion plasmas and dusty laboratory plasmas.

Meanwhile, a growing interest in what is known as quantum plasmas has appeared (see, e.g., [2,3]). Here a main line of research can be found starting from the Schrödinger description of the electron. Assuming that the wave function can be factorized ¹, one may derive a set of fluid equation for the electrons, starting either from an N -body description, a density matrix description, or a Madelung (or Bohm) description of the wave function(s) [2,4]. As in classical fluid mechanics, the set of equations may be closed by a suitable assumption concerning the thermodynamical relation between quantities. These descriptions of the electron fluid, and its interaction with ions and charged dust particles, has been shown to find applications in many different settings [5–14]. Part of the literature has been motivated by recent high field experimental progress and techniques [15–17], or the emergence of new areas, such as spintronics [18].

Indeed, from the experimental perspective, a certain interest has been directed towards the relation of spin properties to the classical theory of motion (see, e.g., [19–31]). In particular, the effects of strong fields on single particles with spin has attracted experimental interest in the laser community [21–26]. However, the main objective of these studies was single particle dynamics, relevant for dilute laboratory systems, whereas our focus will be on collective effects.

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¹There are thus no entanglement properties contained in the model.

Moreover, strong external magnetic fields can be found in astrophysical environments such as pulsar [32,33] and magnetars [34]. Therefore, a great deal of interest has been directed towards finding good descriptions of quantum plasmas in such environments [35–38]. Thus, there is ample need and interest in developing models that are suitable for a wide range of applications, taking into account collective effects in multi-particle systems. There are in the literature several different studies of the effect of spin in plasma systems, e.g. kinetic descriptions in connection to fusion studies [39, 40] and electron spin waves in solid-state plasmas [41]. Moreover, the connection between microscopic and macroscopic spin dynamics in the plasma state has been discussed by de Groot and Suttorp [42].

Inspired by both the historic and recent progress on quantum plasmas, a complete set of multi-fluid spin plasma equations was presented in Ref. [3]. In the current paper, we show, starting from the non-relativistic Pauli equation for spin $\frac{1}{2}$ particles, how a set of plasma equations can be derived for such spin $\frac{1}{2}$ particles. These particles may constitute electrons, positrons (albeit non-relativistic), holes, or similar. Allowing these to interact with ions or charged dust particles, as well as other spin $\frac{1}{2}$ particles, gives the desired governing dynamics of spin plasmas. We furthermore derive the appropriate magnetohydrodynamic description for such quantum plasmas, and investigate the effects of spin on the dynamics of the plasma. The limitations and suitable parameter ranges of the derived governing equations are discussed. The results should be of interest for both laboratory and astrophysical plasmas.

2 Governing equations

In relativistic quantum mechanics, the spin of the electron (and positron) is rigorously introduced through the Dirac Hamiltonian

$$H = c\boldsymbol{\alpha} \cdot \left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right) - e\phi + \beta m_e c^2, \quad (1)$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, e is the magnitude of the electron charge, c is the speed of light, \mathbf{A} is the vector potential, ϕ is the electrostatic potential, and the relevant matrices are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

Here \mathbf{l} is the unit 2×2 matrix and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where we have the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

From the Hamiltonian (1), a nonrelativistic counterpart may be obtained, taking the form

$$H = \frac{1}{2m_e} \left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 + \frac{e\hbar}{2m_e c} \mathbf{B} \cdot \boldsymbol{\sigma} - e\phi. \quad (4)$$

Thus, the electron possesses a magnetic moment $\mathbf{m} = -\mu_B \langle \psi | \boldsymbol{\sigma} | \psi \rangle / \langle \psi | \psi \rangle$, where $\mu_B = e\hbar/2m_e c$ is the Bohr magneton, giving a contribution $-\mathbf{B} \cdot \mathbf{m}$ to the energy. The latter shows the paramagnetic property of the electron, where the spin vector is anti-parallel to the magnetic field in order to minimize the energy of the magnetized system. According to (4) and the relation $dF/dt = \partial F/\partial t + (1/i\hbar)[F, H]$, where F is some operator and $[\cdot, \cdot]$ is the Poisson bracket, we have the following evolution equations for the position and momentum in the Heisenberg picture [43, 44]

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt} = \frac{1}{m_e} \left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right), \quad (5)$$

$$m_e \frac{d\mathbf{v}}{dt} = -e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - \frac{2}{\hbar} \mu_B \nabla (\mathbf{B} \cdot \mathbf{S}), \quad (6)$$

while the spin evolution is given by

$$\frac{d\mathbf{S}}{dt} = \frac{2}{\hbar} \mu_B \mathbf{B} \times \mathbf{S}, \quad (7)$$

where the spin operator is given by

$$\mathbf{S} = \frac{\hbar}{2} \boldsymbol{\sigma}. \quad (8)$$

The above equations thus gives the quantum operator equivalents of the equations of motion for a classical particle, including the evolution of the spin in a magnetic field.

Next, we derive a set of nonlinear fluid equations for a gas of interacting electrons/positrons/holes (for a discussion of a kinetic approach, see Refs. [39] and [40]). The non-relativistic evolution of spin $\frac{1}{2}$ particles, as described by the two-component spinor $\Psi_{(\alpha)}$, is given by (see, e.g. [4])

$$i\hbar \frac{\partial \Psi_{(\alpha)}}{\partial t} = \left[-\frac{\hbar^2}{2m_{j(\alpha)}} \left(\nabla - \frac{iq_{(\alpha)}}{\hbar c} \mathbf{A} \right)^2 - \mu_{(\alpha)} \mathbf{B} \cdot \boldsymbol{\sigma} + q_{(\alpha)} \phi \right] \Psi_{(\alpha)} \quad (9)$$

where $m_{(\alpha)}$ is the particle mass, \mathbf{A} is the vector potential, $q_{(\alpha)}$ the particle charge, $\mu_{(\alpha)}$ the particle's magnetic moment, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli spin matrices, ϕ the electrostatic potential, and (α) enumerates the wave functions. For an electron, the magnetic moment is given by $\mu_e = -\mu_B \equiv -e\hbar/2m_e c$. From now on, we will define $\mu_{(\alpha)} \equiv \mu \equiv q\hbar/2mc$.

Next we introduce the decomposition of the spinors as

$$\Psi_{(\alpha)} = \sqrt{n_{(\alpha)}} \exp(iS_{(\alpha)}/\hbar) \varphi_{(\alpha)}, \quad (10)$$

where $n_{(\alpha)}$ is the density, $S_{(\alpha)}$ is the phase, and $\varphi_{(\alpha)}$ is the 2-spinor through which the spin $\frac{1}{2}$ properties are mediated. Multiplying the Pauli equation (9) by $\Psi_{(\alpha)}^\dagger$, inserting the decomposition (10), and taking the gradient of the resulting phase evolution equation, we obtain the continuity and moment conservation equation

$$\frac{\partial n_{(\alpha)}}{\partial t} + \nabla \cdot (n_{(\alpha)} \mathbf{v}_{(\alpha)}) = 0 \quad (11)$$

and

$$\begin{aligned} m_{(\alpha)} \left(\frac{\partial}{\partial t} + \mathbf{v}_{(\alpha)} \cdot \nabla \right) \mathbf{v}_{(\alpha)} &= q_{(\alpha)} (\mathbf{E} + \mathbf{v}_{(\alpha)} \times \mathbf{B}) \\ &+ \frac{2\mu}{\hbar} (\nabla \otimes \mathbf{B}) \cdot \mathbf{s}_{(\alpha)} - \nabla Q_{(\alpha)} - \frac{1}{m_{(\alpha)} n_{(\alpha)}} \nabla \cdot (n_{(\alpha)} \boldsymbol{\Sigma}_{(\alpha)}) \end{aligned} \quad (12)$$

respectively. The spin contribution to Eq. (12) is consistent with the results of Ref. [42]. Here we have introduced the tensor index $a, b, \dots = 1, 2, 3$, the velocity is defined by

$$\mathbf{v}_{(\alpha)} = \frac{1}{m_{(\alpha)}} \left(\nabla S_{(\alpha)} - i\hbar \varphi_{(\alpha)}^\dagger \nabla \varphi_{(\alpha)} \right) - \frac{q_{(\alpha)}}{m_{(\alpha)} c} \mathbf{A}, \quad (13)$$

the Schrödinger like quantum potential (or Bohm potential) is given by

$$Q_{(\alpha)} = -\frac{\hbar^2}{2m_{(\alpha)} n_{(\alpha)}^{1/2}} \nabla^2 n_{(\alpha)}^{1/2}, \quad (14)$$

the spin density vector is

$$\mathbf{s}_{(\alpha)} = \frac{\hbar}{2} \varphi_{(\alpha)}^\dagger \boldsymbol{\sigma} \varphi_{(\alpha)}, \quad (15)$$

which satisfies $|\mathbf{s}_{(\alpha)}| = \hbar/2$, and we have defined the symmetric gradient spin tensor

$$\boldsymbol{\Sigma}_{(\alpha)} = (\nabla s_{(\alpha)a}) \otimes (\nabla s_{(\alpha)}^a). \quad (16)$$

Moreover, contracting Eq. (9) by $\Psi_{(\alpha)}^\dagger \boldsymbol{\sigma}$, we obtain the spin evolution equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{(\alpha)} \cdot \nabla \right) \mathbf{s}_{(\alpha)} = -\frac{2\mu}{\hbar} \mathbf{B} \times \mathbf{s}_{(\alpha)} + \frac{1}{m_{(\alpha)} n_{(\alpha)}} \mathbf{s}_{(\alpha)} \times [\partial_a (n_{(\alpha)} \partial^a \mathbf{s}_{(\alpha)})]. \quad (17)$$

We note that the particles are coupled via Maxwell's equations.

Suppose that we have N wave functions for the same particle species with mass m , magnetic moment μ , and charge q , and that the total system wave function can be described by the factorization $\Psi = \Psi_{(1)} \Psi_{(2)} \dots \Psi_{(N)}$. Then we define the total particle density for the species with charge q according to

$$n_q = \sum_{(\alpha)=1}^N p_\alpha n_{(\alpha)}, \quad (18)$$

where p_α is the probability related to the wave function $\Psi_{(\alpha)}$. Using the ensemble average $\langle f \rangle = \sum_\alpha p_\alpha (n_{(\alpha)}/n_q) f$ (for any tensorial quantity f), the total fluid velocity for charges q is $\mathbf{V}_q = \langle \mathbf{v}_{(\alpha)} \rangle$ and the total spin density is $\mathbf{S} = \langle \mathbf{s}_{(\alpha)} \rangle$. From these definitions we can define the microscopic velocity in the fluid rest frame according to $\mathbf{w}_{(\alpha)} = \mathbf{v}_{(\alpha)} - \mathbf{V}$, satisfying $\langle \mathbf{w}_{(\alpha)} \rangle = 0$, and the microscopic spin density $\mathbf{S}_{(\alpha)} = \mathbf{s}_{(\alpha)} - \mathbf{S}$, such that $\langle \mathbf{S}_{(\alpha)} \rangle = 0$.

Taking the ensemble average of Eqs. (11), (12), and (17), we obtain

$$\frac{\partial n_q}{\partial t} + \nabla \cdot (n_q \mathbf{V}_q) = 0, \quad (19)$$

$$m n_q \left(\frac{\partial}{\partial t} + \mathbf{V}_q \cdot \nabla \right) \mathbf{V}_q = q n_q (\mathbf{E} + \mathbf{V}_q \times \mathbf{B}) - \nabla \cdot \boldsymbol{\Pi}_q - \nabla P_q + \mathcal{C}_{qi} + \mathbf{F}_Q \quad (20)$$

and

$$n_q \left(\frac{\partial}{\partial t} + \mathbf{V}_q \cdot \nabla \right) \mathbf{S} = -\frac{2\mu n_q}{\hbar} \mathbf{B} \times \mathbf{S} - \nabla \cdot \boldsymbol{\mathcal{K}}_q + \boldsymbol{\Omega}_S \quad (21)$$

respectively. Here we have added the collisions \mathcal{C}_{qi} between charges q and the ions i , denoted the total quantum force density by

$$\begin{aligned} \mathbf{F}_Q = & \frac{2\mu n_q}{\hbar} (\nabla \otimes \mathbf{B}) \cdot \mathbf{S} - n_q \langle \nabla Q_{(\alpha)} \rangle - \frac{1}{m} \nabla \cdot (n_q \boldsymbol{\Sigma}) - \frac{1}{m} \nabla \cdot (n_q \tilde{\boldsymbol{\Sigma}}) \\ & - \frac{1}{m} \nabla \cdot [n_q (\nabla S_a) \otimes \langle (\nabla S_{(\alpha)}^a) \rangle + n_q \langle (\nabla S_{(\alpha)a}) \rangle \otimes (\nabla S^a)], \end{aligned} \quad (22)$$

consistent with the results in Ref. [42], and defined the the nonlinear spin fluid correction according to

$$\begin{aligned} \boldsymbol{\Omega}_S = & \frac{1}{m} \mathbf{S} \times [\partial_a (n_q \partial^a \mathbf{S})] + \frac{1}{m} \mathbf{S} \times [\partial_a (n_q \langle \partial^a \mathbf{S}_{(\alpha)} \rangle)] \\ & + \frac{n_q}{m} \left\langle \frac{\mathbf{S}_{(\alpha)}}{n_{(\alpha)}} \times [\partial_a (n_{(\alpha)}) \partial^a \mathbf{S}] \right\rangle + \frac{n_q}{m} \left\langle \frac{\mathbf{S}_{(\alpha)}}{n_{(\alpha)}} \times [\partial_a (n_{(\alpha)} \partial^a \mathbf{S}_{(\alpha)})] \right\rangle, \end{aligned} \quad (23)$$

where $\mathbf{\Pi} = mn[\langle \mathbf{w}_{(\alpha)} \otimes \mathbf{w}_{(\alpha)} \rangle - \mathbf{I} \langle w_{(\alpha)}^2 \rangle / 3]$ is the trace-free anisotropic pressure tensor (\mathbf{I} is the unit tensor), $P = mn \langle w_{(\alpha)}^2 \rangle$ is the isotropic scalar pressure, $\mathbf{\Sigma} = (\nabla S_a) \otimes (\nabla S^a)$ is the nonlinear spin correction to the classical momentum equation, $\tilde{\mathbf{\Sigma}} = \langle (\nabla S_{(\alpha)a}) \otimes (\nabla S_{(\alpha)}^a) \rangle$ is a pressure like spin term (which may be decomposed into trace-free part and trace), $\mathbf{K} = n \langle \mathbf{w}_{(\alpha)} \otimes \mathbf{S}_{(\alpha)} \rangle$ is the thermal-spin coupling, and $[(\nabla \otimes \mathbf{B}) \cdot \mathbf{S}]^a = (\partial^a B_b) S^b$. Here the indices $a, b, \dots = 1, 2, 3$ denotes the Cartesian components of the corresponding tensor. We note that the momentum conservation equation (20) and the spin evolution equation (21) still contains the explicit sum over the N states.

The coupling between the quantum plasma species is mediated by the electromagnetic field. By definition, we let $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ where $\mathbf{M} = 2n_q \mu \mathbf{S}/\hbar$ is the magnetization due to the spin sources. Ampère's law $\nabla \times \mathbf{H} = \mathbf{j} + \epsilon_0 \partial_t \mathbf{E}$ then takes the form

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \mathbf{j}_M) + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (24)$$

where we have the magnetization spin current $\mathbf{j}_M = \nabla \times \mathbf{M}$ and the free current \mathbf{j} . The system is closed by Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (25)$$

3 Electron-ion plasma and the magnetohydrodynamic limit

The preceding analysis applies equally well to electrons as holes or similar condensations. We will now assume that the quantum particles are electrons, thus $q = -e$, where e is the magnitude of the electron charge. By the inclusion of the ion species, which are assumed to be described by the classical equations and have charge Ze , we may derive a set of one-fluid equations. The ion equations read

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{V}_i) = 0, \quad (26)$$

and

$$m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla \right) \mathbf{V}_i = Z e n_i (\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_i - \nabla P_i + \mathcal{C}_{iq}. \quad (27)$$

Next we define the total mass density $\rho \equiv (m_e n_e + m_i n_i)$, the centre-of-mass fluid flow velocity $\mathbf{V} \equiv (m_e n_e \mathbf{V}_e + m_i n_i \mathbf{V}_i)/\rho$, and the current density $\mathbf{j} = -e n_e \mathbf{V}_e + Z e n_i \mathbf{V}_i$. Using these definitions, we immediately obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (28)$$

from Eqs. (19) and (26). Assuming quasi-neutrality, i.e. $n_e \approx Z n_i$, the momentum conservation equations (20) and (27) give

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{j} \times \mathbf{B} - \nabla \cdot \mathbf{\Pi} - \nabla P + \mathbf{F}_Q, \quad (29)$$

where $\mathbf{\Pi}$ is the tracefree pressure tensor in the centre-of-mass frame, P is the scalar pressure in the centre-of-mass frame, and the collisional contributions cancel due to momentum conservation. We also note that due to quasi-neutrality, we have $n_e = \rho/(m_e + m_i/Z)$ and $\mathbf{V}_e = \mathbf{V} - m_i \mathbf{j}/Z e \rho$,

and we can thus express the quantum terms in terms of the total mass density ρ , the centre-of-mass fluid velocity \mathbf{V} , and the current \mathbf{j} . With this, the spin transport equation (21) reads

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{S} = \frac{m_e}{Ze} \mathbf{j} \cdot \nabla \mathbf{S} - \frac{2\mu\rho}{\hbar} \mathbf{B} \times \mathbf{S} - \left(m_e + \frac{m_i}{Z} \right) \nabla \cdot \mathbf{K}_q + \left(m_e + \frac{m_i}{Z} \right) \Omega_S. \quad (30)$$

In the momentum equation (29), we have the force density $\mathbf{j} \times \mathbf{B} + \mathbf{F}_Q$, where \mathbf{j} is the current produced by the free charges. In general, for a magnetized medium with magnetization density \mathbf{M} , Ampère's law gives the free current in a finite volume V according to

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \nabla \times \mathbf{M}, \quad (31)$$

where we have neglected the displacement current is $\mathbf{j}_D = \epsilon_0 \partial_t \mathbf{E}$. The delta-function contribution of the surface current is an important part of the total current when we are interested in the forces on a finite volume, as will now be shown.

It is worth noting that the expression of the force density in the momentum conservation equation can, to lowest order in the spin, be derived on general macroscopic grounds. Formally, the total force density on a volume element V is defined as $\mathbf{F} = \lim_{V \rightarrow 0} (\sum_\alpha \mathbf{f}_\alpha / V)$, where \mathbf{f}_α are the different forces acting on the volume element, and might include surface forces as well. For magnetized matter, the total force on an element of volume V is then

$$\mathbf{f}_{\text{tot}} = \int_V \mathbf{j}_{\text{tot}} \times \mathbf{B} dV + \oint_{\partial V} (\mathbf{M} \times \hat{\mathbf{n}}) \times \mathbf{B} dS \quad (32)$$

where (neglecting the displacement current) $\mathbf{j}_{\text{tot}} = \mathbf{j} + \nabla \times \mathbf{M}$. Inserting the expression for the total current into the volume integral and using the divergence theorem on the surface integral, we obtain the force density

$$\mathbf{F}_{\text{tot}} = \mathbf{j} \times \mathbf{B} + M_k \nabla B^k, \quad (33)$$

identical to the lowest order description from the Pauli equation (see Eq. (29)). Inserting the free current expression (31), due to Ampère's law, we can write the total force density according to

$$F^i = -\partial^i \left(\frac{B^2}{2\mu_0} - \mathbf{M} \cdot \mathbf{B} \right) + \partial_k (H^i B^k). \quad (34)$$

The first gradient term in Eq. (34) can be interpreted as the force due to a potential (the energy of the magnetic field and the magnetization vector in that field), while the second divergence term is the anisotropic magnetic pressure effect. Noting that the spatial part of the stress tensor takes the form $T^{ik} = -H^i B^k + (B^2/2\mu_0 - \mathbf{M} \cdot \mathbf{B})\delta^{ik}$ [42], we see that the total force density on the magnetized fluid element can be written $F^i = -\partial_k T^{ik}$, as expected. Thus, the Pauli theory results in the same type of conservation laws as the macroscopic theory. The momentum conservation equation (29) then reads

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = -\nabla \left(\frac{B^2}{2\mu_0} - \mathbf{M} \cdot \mathbf{B} \right) + B^k \partial_k \mathbf{H} - \nabla P, \quad (35)$$

where for the sake of clarity we have assumed an isotropic pressure, dropped the displacement current term in accordance with the nonrelativistic assumption, and neglected the Bohm potential (these terms can of course simply be added to (35)).

Approximating the quantum corrections, using $L \gg \lambda_F$ where L is the typical fluid length scale and λ_F is the Fermi wavelength for the particles, according to [2]

$$\langle \nabla Q_{(\alpha)} \rangle \approx -\nabla \left(\frac{\hbar^2}{2mn_q^{1/2}} \nabla^2 n_q^{1/2} \right) \equiv \nabla Q. \quad (36)$$

We then note that even if Q is small, the magnetic field may, through the dynamo equation (25), still be driven by pure quantum effects through the spin.

A generalized Ohm's law may be derived assuming $\mathcal{C}_{ei} = en_e \eta \mathbf{j}$, where η is the resistivity. From the electron momentum conservation equation (20) combined with Faraday's law we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left\{ \mathbf{V} \times \mathbf{B} - \frac{\mathbf{j} \times \mathbf{B}}{en_e} - \eta \mathbf{j} + \frac{m_e}{e} \frac{d\mathbf{V}_e}{dt} - \frac{\mathbf{F}_Q}{en_e} \right\} \quad (37)$$

where η is the resistivity. Here we have omitted the anisotropic part of the pressure, and neglected terms of order m_e/m_i compared with unity.

The electron inertia term is negligible unless the electron velocity is much larger than the ion velocity. Thus whenever electron inertia is important, we include only the electron contribution to the current, and use Ampère's law to substitute $\mathbf{V}_e = \nabla \times \mathbf{B} / en_e \mu_0$ into the term proportional to m_e in (37), which gives the final form of the Generalized Ohm's law

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left\{ \mathbf{V} \times \mathbf{B} - \frac{\mathbf{j} \times \mathbf{B}}{en_e} - \eta \mathbf{j} - \frac{m_e}{e^2 \mu_0} \left[\frac{\partial}{\partial t} - \left(\frac{\nabla \times \mathbf{B}}{e \mu_0 n_e} \right) \cdot \nabla \right] \frac{\nabla \times \mathbf{B}}{n_e} - \frac{\mathbf{F}_Q}{en_e} \right\} \quad (38)$$

In the standard MHD regime the Hall term and the electron inertia term are negligible. During such conditions the quantum force is also negligible in Ohm's law, which reduces to its standard MHD form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B} - \eta \mathbf{j}) \quad (39)$$

Note, however, that the quantum force including the spin effects still should be kept in the momentum equation (29). Equations (28), (29), and (38) together with the spin evolution equation (30), which is needed to determine \mathbf{F}_Q , constitutes the basic equations. In order to close the system, equations of state for the pressure as well as for the spin state are needed, as will be discussed in section 5.

4 Closing the system

The momentum equation, Ohm's (generalized) law and the continuity equation need to be completed by an equation of state for the pressure. As is well-known, rigorous derivations of the equation of state is only applicable in special cases of limited applicability to real plasmas. Useful models include a scalar pressure where the pressure is proportional to some power of the density, i.e.

$$\frac{d}{dt} \left(\frac{P_s}{n_s^{\gamma_s}} \right) = 0, \quad (40)$$

where $d/dt = \partial/\partial t + \mathbf{V}_s \cdot \nabla$ is the convective derivative and γ_s is the ratio of specific heats, which in general can be different for different species s . Secondly, we note that the magnitude of the terms that are quadratic in the spin depends highly on the spatial scale of the variations. In MHD, the scale lengths are typically longer or equal to the Larmor radius of the heavier particles, which means that the terms that are quadratic in S can be neglected in the expression for the quantum force \mathbf{F}_Q as well as in the spin evolution equation (30). To lowest order, the spin inertia can be neglected for frequencies well below the electron cyclotron frequency. Also omitting the spin-thermal coupling term, which is small for the same reasons as stated above, the spin-vector is determined from

$$\mathbf{B} \times \mathbf{S} = 0. \quad (41)$$

which has a solution

$$\mathbf{S} = -\frac{\hbar}{2} \eta\left(\frac{\mu_B B}{k_B T_e}\right) \hat{\mathbf{B}} \quad (42)$$

consistent with standard theories of paramagnetism, as the spin anti-parallel to the magnetic field minimizes the magnetic moment energy, so that

$$\mathbf{M} = 2\mu_B n_e \eta\left(\frac{\mu_B B}{k_B T_e}\right) \hat{\mathbf{B}}. \quad (43)$$

Here B denotes the magnitude of the magnetic field, $\eta(x) = \tanh x$ is the Brillouin function,² and $\hat{\mathbf{B}}$ is a unit vector in the direction of the magnetic field. In this approximation the spin evolution equation (30) can be dropped, and the quantum force can be written

$$\mathbf{F}_Q = n_e \nabla \left(\frac{\hbar^2}{2mn_e^{1/2}} \nabla^2 n_e^{1/2} \right) + 2n_e \eta\left(\frac{\mu_B B}{k_B T_e}\right) \mu_B \nabla B \quad (44)$$

where the second term comes from the spin. Combining the approximations presented in this section together with the MHD equations presented in section 3, or either of the two dust systems presented in section 4, closed systems are obtained.

5 Illustrative example

Let us next consider the spin-model described by (28), (35), and (39) with the resistivity put to zero. Furthermore, we note that for most plasmas $\mu_B B \ll k_B T$, and thus we can use the approximation $\eta(\mu_B B/k_B T) \approx \mu_B B/k_B T$. For definiteness, we also choose an isothermal equation of state such that $\nabla P = k_B T \nabla n$. Next we let $\mathbf{B} = B_0 \hat{\mathbf{z}} + \mathbf{B}_1$, $\mathbf{M} = M_0 \hat{\mathbf{z}} + \mathbf{M}_1$, $n = n_0 + n_1$, where index 0 denotes the equilibrium part and index 1 denote the perturbation, and we have assumed that the equilibrium part of the velocity is zero. Linearizing around the equilibrium and Fourier analysing, and omitting the non-spin part of the quantum force,³ we find that the general dispersion relation can be written

$$\left(\omega^2 - k^2 \tilde{C}_A^2 \right) \left[\left(\omega^2 - k^2 \tilde{C}_A^2 + k_x^2 \tilde{c}_s^2 \right) \left(\omega^2 - k_z^2 \tilde{c}_s^2 \right) + k_x^2 k_z^2 \tilde{c}_s^4 \right] = 0 \quad (45)$$

where $c_s = [(k_B T_e + k_B T_i)/m_i]^{1/2}$ is the standard acoustic velocity. Formally, Eq. (45) looks similar to the standard ideal MHD dispersion relation. However, we note that firstly spin effects appear in the spin-modified Alfvén velocity

$$\tilde{C}_A = C_A \left(1 - \frac{\hbar^2 \omega_{pe}^2}{mc^2 k_B T} \right)^{1/2} \quad (46)$$

where ω_{pe} is the electron plasma frequency and $C_A = (B_0^2/\mu_0 \rho_0)^{1/2}$ is the standard Alfvén velocity. Moreover, another spin-modification is introduced in the factor \tilde{c}_s , given by

$$\tilde{c}_s = c_s \left[1 - \left(\frac{\hbar \omega_{ce}}{k_B T} \right)^2 \right]^{1/2} \quad (47)$$

²In general, the thermodynamic equilibrium distribution of spin results in a magnetization proportional to the Brillouin function $B_j(\mu_B/k_B T)$, where j is the spin of the particles in question. For the special case of spin 1/2-particles we have $B_{1/2}(x) = \tanh(x)$.

³The ordinary part of the quantum force is negligible compared to the spin coupling provided $eB_0 \gg \hbar k^2$.

The first factor of Eq. (45) describes the spin-modified shear Alfvén waves, and the second factor describes fast and slow magnetosonic waves. We point out that the assumption made in this section initially, $\mu_B B \ll k_B T$, implies that both \tilde{C}_A and \tilde{c}_s given by (46) and (47) respectively are real. Moreover, we stress that we cannot simply obtain the spin-modified magnetosonic dispersion relation from the classical one by the substitutions $C_A \rightarrow \tilde{C}_A$ and $c_s \rightarrow \tilde{c}_s$, since the classical acoustic velocity c_s still appears in the factor $(\omega^2 - k_z^2 c_s^2)$, as seen from (45). We end this section by noting that the modification of the Alfvén velocity can be appreciable for a high density low temperature plasma.

6 Summary and Discussion

In the present paper we have derived one-Fluid MHD equations for a number of different plasmas including the effects of the electron spin, starting from the Pauli equation for the individual particles. In particular we have derived spin-MHD equations for an electron-ion plasma. Furthermore, the general equations derived in section 2 constitutes a basis for an electron-positron plasma description including spin effects.

In order to obtain closure of the system, our equations needs to be supplemented by equations of state for the pressure as well as for the spin pressure. In the MHD regime, a rather simple way to achieve this closure has been discussed in Section 4, where we assume that the scale lengths are long enough such terms that are quadratic in the spin vector as well as the spin-thermal coupling are neglected. However, we here note that if more elaborate models are used, the spin pressure together with the spin-thermal coupling might play an important role in the generalized Ohms's law (38).

Since the spin-coupling give raise to a parallel force (to the magnetic field) in the momentum equation, the parallel electric field will not be completely shielded even for zero temperature, contrary to ordinary MHD. As an immediate consequence, the spin-coupling can give rise to a rich variety of physical effects. In this paper we have limited ourselves to present a single example, linear wave propagation in an electron-ion plasma, and shown the modification of the dispersion relation due to the spin effects. In particular, we note that the spin effects are important for low temperature, high densities, and/or strongly magnetized plasmas. The latter can be found in astrophysical systems, such as pulsars or magnetars [38], where however other modifications, such as vacuum polarization and magnetization [15, 45], of our set of governing equations may be necessary in order to accurately model such plasmas. Moreover, ultra-cold plasmas may even be found in laboratory environments, where currently mK temperature plasmas can be formed [46]. Studies involving the spin dynamics through the spin evolution equation (21), kinetic effects associated with the spin, as well as nonlinear spin dynamics are projects for future work. Finally, we note that dusty plasmas can sustain weakly damped modes with low phase velocities [47], and quantum and spin effects tend to be important in this regime.

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